

REDUCED RELATIVE TUTTE, KAUFFMAN BRACKET AND JONES POLYNOMIALS OF VIRTUAL LINK FAMILIES

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Abstract

This paper contains general formulas for the reduced relative Tutte, Kauffman bracket and Jones polynomials of families of virtual knots and links given in Conway notation and a counterexample to Z -move conjecture.

1. Introduction

According to Thistlethwaite's Theorem [1], the Jones polynomial and Kauffman bracket polynomial of a knot or link (shortly KL) can be computed from the (signed) graph of the KL [2] via the Tutte polynomial. Without computing Tutte polynomials, the Jones polynomials of the diagrams with at most 5 twist regions in which crossings can be inserted are computed by D. Silver, A. Stoimenow, and S. Williams [3], and the general method for computing Kauffman bracket polynomial of such diagrams is outlined by A. Champanerkar and I. Kofman [4].

The general formulas for Tutte and Jones polynomials for families of classical KL s with at most 5 twisting regions, given in Conway notation, are derived in [5].

Knots and links (or shortly KL s) can be given in Conway notation [6, 7, 8, 9]. Analogous to the Conway notation for classical KL s we use extended Conway notation for virtual KL s. The extended Conway notation for virtual links differs from the standard one in the following way:

- sequence of n classical vertices $1, \dots, 1$ is denoted by 1^n
- sequence of n classical negative vertices $-1, \dots, -1$ by $(-1)^n$
- virtual vertices are denoted by i .

The relative Tutte polynomial of colored graphs is introduced in the paper [10] by Y. Diao and G. Hetei. An alternative approach to the computation of Bollobás-Riordan polynomial and Kauffman bracket polynomial of virtual KL s, via ribbon graphs, is given by S. Chmutov and I. Pak [11], and S. Chmutov [12].

Zeroes of the Jones polynomials of different knots and their plots are computed in [13, 14, 15, 16]. Zeroes of Jones polynomials corresponding to different families of classical (alternating and non-alternating) KL s, called "portraits of the families", are given in [5].

First we restrict our attention to graphs corresponding only to alternating virtual KL s, hence we consider graphs corresponding to virtual KL s with all edges labeled by $+$ or 0 (see [4], Section 5), with variables $X_+ = X$, $Y_+ = Y$, $x_+ = x$, $y_+ = y$. In this setting, we have the following recursive formula for computing the relative Tutte polynomial:

$$T(G) = \begin{cases} XT(G/e) & \text{if } e \text{ is a bridge} & (1) \\ YT(G-e) & \text{if } e \text{ is a loop} & (2) \\ xT(G-e) + yT(G/e) & \text{if } e \text{ is neither a loop or a bridge} & (3) \end{cases}$$

where $G - e$ is the graph obtained from G by deleting the edge e , and G/e is the graph obtained from G by contracting e .

Notice that formulas for the relative Tutte polynomial of non-alternating virtual KL s, can be obtained from general formulas for alternating virtual KL s by substituting negative values of parameters.

According to the Theorem 5.4 [10], the relative Tutte polynomial $T_H(G)$ (see Section 5.1 [10]) corresponding to a virtual link diagram K gives the Kauffman bracket through the following variable substitution:

$$X \rightarrow -A^{-3}, \quad Y \rightarrow -A^3, \quad x \rightarrow A, \quad y \rightarrow A^{-1}, \quad d \rightarrow -(A^2 + A^{-2}),$$

and the Jones polynomial of K is obtained from the Kauffman bracket polynomial by substituting $A = t^{-\frac{1}{4}}$.

2. Reduction rules

Z -move is equivalent to the virtualization [17]. Since the Kauffman bracket and Jones polynomial of a virtual link L are invariant under classical and virtual Reidemeister move II and Z -move ([17], Lemma 7) on graphs, we can simplify graphs before computations. Graphs of virtual KL s can be simplified using the following reduction rules:

1. Reidemeister move II for classical crossings (Fig. 1a);

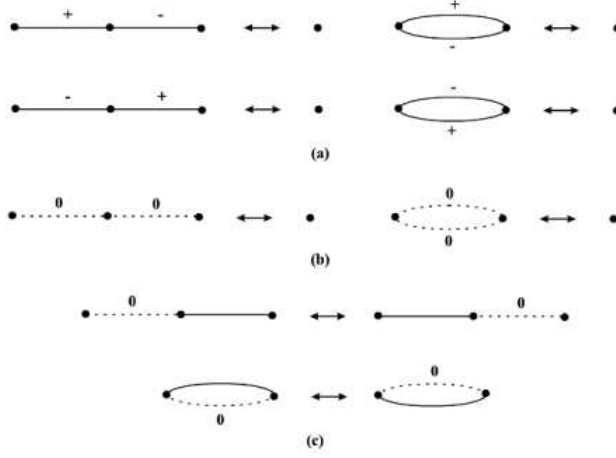


Figure 1: (a) Reidemeister move II for classical crossings; (b) Reidemeister move II for virtual crossings; (c) Z -move on KL graphs.

2. Reidemeister move II for virtual crossings (Fig. 1b);
3. Z -move (Fig. 1c) on KL graphs.

where 0-edges are denoted by broken lines. The relative Tutte polynomial obtained after this reduction will be called *reduced relative Tutte polynomial*.

Graphs obtained by reductions can contain only cycles, or multiple edges of the same sign, and at most one 0-edge. A graph is called completely reduced if none of the reduction rules can be applied further.

2.1. Family \mathbf{p}

Consider the family \mathbf{p} ($p \geq 2$), which consists of classical knots and links of the form 1^p . Graphs corresponding to links of this family are cycles of length p , satisfying the following recursion:

$$T(G(1^p)) = yX^{p-1} + xT(G(1^{p-1}))$$

with $T(G(1^2)) = yX + xY$, so the general formula for the reduced relative Tutte polynomial of the graph $G((1^p))$ is

$$T(G(1^p)) = \frac{(x^p - X^p)}{x - X}y + (Y - y)x^{p-1}.$$

Virtual KL s of the form $(i, 1^p)$ which belong to the same family, correspond to the reduced graphs shown on Fig. 2, and their Tutte polynomials satisfy the following recursion:

$$T(G(i, 1^p)) = yX^{p-1} + xT(G(i, 1^{p-1}))$$

with $T(G(i, 1)) = x + y$, so the general formula for the reduced relative Tutte polynomial of the graph $G((i, 1^p))$ is

$$T(G(i, 1^p)) = x^p + \frac{(x^p - X^p)}{x - X}y.$$

As a corollary of this general formula we obtain reduced relative Tutte polynomials for positive or negative values of the parameter p , and Kauffman bracket and Jones polynomials. For example, for $p = 2$ we obtain the reduced relative Tutte polynomial $T(G(i, 1^2)) = x^2 + xy + Xy$ of the virtual trefoil $(i, 1, 1)$, and for $p = -2$ the reduced relative Tutte polynomial $T(G(i, (-1)^2)) = \frac{1}{x^2} - \frac{y}{xX^2} - \frac{y}{x^2X}$ of its mirror image $(i, -1, -1)$.

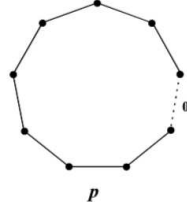


Figure 2: Graph $G(i, 1^p)$.

2.2. Family $p q$

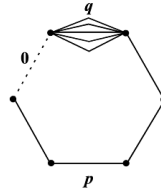


Figure 3: Graph $G((i, 1^p)(1^q))$.

Next we consider virtual KL s of the form $(i, 1^p)(1^q)$ in link family $\mathbf{p q}$. The reduced relative Tutte polynomials of their graphs (Fig. 3) satisfy the recursion:

$$T(G(i, 1^p)) = yX^{p-1}T(\overline{G}(1^q)) + xT(G((i, 1^{p-1})(1^q)))$$

where $\overline{G}(1^q)$ is the dual of the graph $G(1^q)$, and $T(G((i, 1)(1))) = xy + x^2y + x^3$, so the general formula for the reduced relative Tutte polynomial of the graph $G((i, 1^p)(1^q))$ is

$$T(G((i, 1^p) 1(1^q))) = \frac{(x^{q+1} - X^{q+1})(y^{p+1} - Y^{p+1})}{(x - X)(y - Y)}$$

$$- \frac{(y^{p+1} - Y^{p+1})}{y - Y} X^q - \frac{(x^{q+1} - X^{q+1})}{x - X} Y^n + \frac{(x^q - X^q)}{x - X} y^{n+1} + X^q Y^n - x^q Y^n.$$

2.3. Family $p \ 1 \ q$

Based on the link family $p \ 1 \ q$ we construct three families of virtual KL s with different reduced graphs: $(i, 1^p) 1(1^q)$, $(i, 1^p) 1(i, 1^q)$, and $(1^p) i(1^q)$.

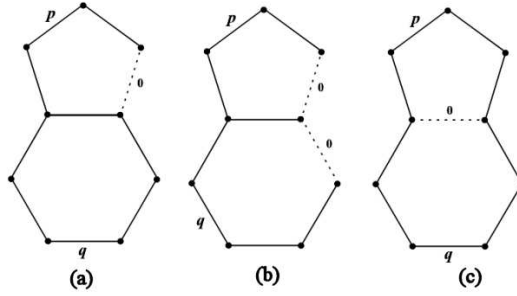


Figure 4: Graphs (a) $G((i, 1^p) 1(1^q))$; (b) $G((i, 1^p) 1(i, 1^q))$, and (c) $G((1^p) i(1^q))$.

The reduced relative Tutte polynomials of the graphs corresponding to the family $(i, 1^p) 1(1^q)$ (Fig. 4a) satisfy the relation:

$$T(G((i, 1^p) 1(1^q))) = yT(G((i, 1^{p+q})) + xT(G((i, 1^p)))T(G((1^q))), \quad (2.1)$$

and the reduced relative Tutte polynomials of the graphs corresponding to the family $(i, 1^p) 1(i, 1^q)$ (Fig. 4b) satisfy the relation:

$$T(G((i, 1^p) 1(i, 1^q))) = yT(G((1^{p+q})) + xT(G((i, 1^p)))T(G((i, 1^q))), \quad (2.2)$$

so the general formulas for their reduced relative Tutte polynomials can be derived from the previous formulas (2.2).

The relative Tutte polynomial of the reduced graphs corresponding to the family $(1^p) i(1^q)$ (Fig. 4c) satisfy the recursion:

$$T(G((1^p) i(1^q))) = yX^{q-1}T(G((i, 1^p))) + xT(G((1^p) i(1^{q-1}))), \quad (2.3)$$

with $T(G((1^p) i 1)) = T(G((i, 1^p) 1))$.

Hence, the general formula for the reduced relative Tutte polynomial of the graph $G((1^p) i(1^q))$ is

$$\begin{aligned}
T(G((1^p) i (1^q))) &= \frac{(x^p - X^p)}{x - X}(x + X)x^{q-1}y + \frac{(x^{q-p-1} - X^{q-p-1})}{x - X}x^pX^{p+1}y \\
&+ \frac{(x^{p-2} - X^{p-2})}{x - X}\frac{(x^q - X^q)}{x - X}x^2y^2 + \frac{(x^q - X^q)}{x - X}(x + X)X^{p-2}y^2 + x^{p+q-1}Y.
\end{aligned}$$

Same as before, from the general formula for reduced relative Tutte polynomials with negative values of parameters we obtain reduced relative Tutte polynomials expressed as Laurent polynomials. For example, for $p = 4$, $q = -3$ from the preceding general formula we obtain reduced relative Tutte polynomial of the virtual knot $(1, 1, 1, 1) i (-1, -1, -1)$:

$$\frac{-x^3y}{X^3} - \frac{x^2y}{X^2} - \frac{xy}{X} + \frac{Xy}{x} + \frac{X^2y}{x^2} + \frac{X^3y}{x^3} - 3\frac{y^2}{x} - \frac{x^2y^2}{X^3} - 2\frac{xy^2}{X^2} - 3\frac{y^2}{X} - 2\frac{Xy^2}{x^2} - \frac{X^2y^2}{x^3} + Y$$

and for $p = -4$, $q = 3$ we obtain reduced relative Tutte polynomial of its mirror image $(-1, -1, -1, -1) i (1, 1, 1)$:

$$\frac{x^2y}{X^4} - \frac{xy}{X^3} - \frac{y}{X^2} - \frac{y}{xX} + \frac{Xy}{x^3} + \frac{X^2y}{x^4} - 2\frac{y^2}{x^3} - \frac{xy^2}{X^4} - 2\frac{y^2}{X^3} - 3\frac{y^2}{xX^2} - 3\frac{y^2}{x^2X} - \frac{Xy^2}{x^4} + \frac{Y}{x^2}.$$

Appropriate substitutions of the variables, yield their Kauffman bracket and Jones polynomials.

3. Virtual knots with trivial Jones polynomial

For classical knots, the question whether non-trivial knot with unit Jones polynomial exists is still open. In the case of virtual knots, it is easy to make infinitely many non-trivial virtual knots with unit Jones polynomial [17]. Among 2171 prime virtual knots derived from classical knots with at most $n = 8$ crossings, there are 272 knots (about 12%) with unit Jones and Kauffman bracket polynomial. The smallest virtual knot with unit Jones polynomial is $1, 1, i, -1, i$ ([17], Fig. 17), that can be simply generalized to an infinite family of different non-trivial virtual knots of the form $1^p, i, (-1)^q, i, p - q = 1$, with unit Jones polynomial (where 1^p denotes a sequence $1, \dots, 1$ of the length p , and -1^q a sequence $-1, \dots, -1$ of the length q). In the same way, for $p - q = 2k + 1$, ($k \geq 1$) all mutually different virtual knots $1^p, i, (-1)^q, i$ will have the same Jones polynomial as the classical knots $(2k + 1)_1$ ($3_1, 5_1, 7_1, \dots$, or $3, 5, 7, \dots$ in Conway notation), respectively.

For virtual KL s that can be reduced to unknot (unlink) by a series of Reidemeister moves for virtual knots and Z -moves we will say that they are Z -move equivalent to the unknot (unlink). The most of mentioned 272 knots with unit Jones polynomial are Z -move equivalent to the unknot. Hence, R. Fenn, L.H. Kauffman, and V.O. Manturov [23] proposed Z -move conjecture:

Conjecture 3.1 *Every knot with unit Jones polynomial is Z -move equivalent to the unknot.*

The main candidates for counterexamples to this conjecture have been virtual knots $KS = (((1, (i, 1), -1), -1), i, 1)$ (Fig. 5a,b) and $S7 = (i, 1) 1, (i, -1) - 1, (i, 1)$ (Fig. 6). For each of them, non-triviality can be proved in many different ways (e.g., for the first of them by parity arguments, or by computing their other polynomial invariants: Sawollek polynomial, Miyazawa polynomials [19], Kauffman arrow polynomial [18], or 2-cabled Jones polynomial. It is interesting to notice that the knot KS has unit Miyazawa polynomials as well.

After proving that the virtual knot KS with unit Jones polynomial is not Z -move equivalent to unknot, this conjecture is now known to be false (see [20, 25]). Methods for proving that counterexamples that we discuss below are indeed counterexamples depend on the use of parity in virtual knot theory as introduced in [20, 24, 25]. See the next section for a detailed discussion of this method.

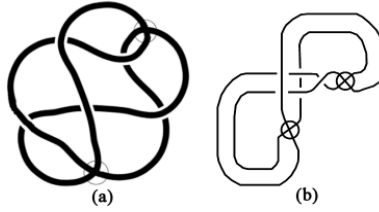


Figure 5: (a),(b) Virtual knot $KS = (((1, (i, 1), -1), -1), i, 1)$.

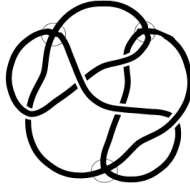


Figure 6: Virtual knot $S7 = (i, 1) 1, (i, -1) - 1, (i, 1)$.

Among virtual knots with unit Jones polynomial, probably the most interesting is the virtual knot $(1, i, -1) i (1, i, -1)$ (Fig. 7a), which generates the family of virtual knots $(1^k, i, (-1)^k) i (1^k, i, (-1)^k)$ ($k \geq 1$) (Fig. 7b) which cannot be distinguished from unknot by any of the mentioned polynomial invariants, except 2-colored Jones polynomial. All members of this family are Z -move equivalent to the unknot.

Another interesting virtual knot is $(i, 1) (1, i) 1 (-1, i)$ which has all trivial mentioned polynomial invariants, except 2-colored Jones polynomial. Moreover, it is another candidate for a counterexample to Z -move conjecture (Fig. 8).

Using Z -move reduction we can prove many simple facts about Jones polynomial of virtual KL s. For example:

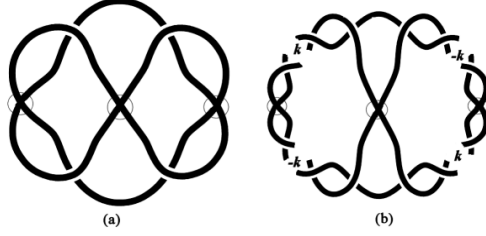


Figure 7: (a) Virtual knot $(1, i, -1) i (1, i, -1)$; (b) family $(1^k, i, (-1)^k) i (1^k, i, (-1)^k)$.

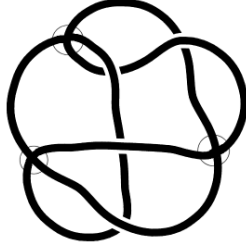


Figure 8: Virtual knot $(i, 1) (1, i) 1 (-1, i)$.

- if every twist of a knot or link L which contains virtual crossings has an even number of them, the Jones polynomial of L is equal to the Jones polynomial of the classical link L' obtained from L by deleting virtual crossings.
- Let be given an alternating knot K with n crossings. In order to unknot it, in any its minimal diagram it is sufficient to make at most $m = \lfloor \frac{n}{2} \rfloor$ crossing changes. If a minimal diagram D of K can be unknotted by the crossing changes in the crossings C_1, \dots, C_k ($k \leq m$), the virtual knot diagram D' , obtained from D by substituting every positive crossing C_i ($i \in \{1, \dots, k\}$) by $i, -1, i$, and every negative crossing C_j ($j \in \{1, \dots, k\}$) by $i, 1, i$, has unit Jones polynomial. Are all virtual knots obtained in this way non-trivial? Can we obtain from two different minimal diagrams of K two different virtual knots?

4. The Parity Bracket Polynomial

In this section we introduce the Parity Bracket Polynomial of Vassily Manturov [20]. This is a generalization of the bracket polynomial to virtual knots and links that uses the parity of the crossings. We define a *Parity State* of a virtual diagram K to be a labeled virtual graph obtained from K as follows: For each odd crossing in K replace the crossing by a graphical node. For each even crossing in K replace

the crossing by one of its two possible smoothings, and label the smoothing site by A or A^{-1} in the usual way. Then we define the parity bracket by the state expansion formula

$$\langle K \rangle_P = \sum_S A^{n(S)} [S]$$

where $n(S)$ denotes the number of A -smoothings minus the number of A^{-1} smoothings and $[S]$ denotes a combinatorial evaluation of the state defined as follows: First reduce the state by Reidemeister two moves on nodes as shown in Figure 9. The graphs are taken up to virtual equivalence (planar isotopy plus detour moves on the virtual crossings). Then regard the reduced state as a disjoint union of standard state loops (without nodes) and graphs that irreducibly contain nodes. With this we write

$$[S] = (-A^2 - A^{-2})^{l(S)} [G(S)]$$

where $l(S)$ is the number of standard loops in the reduction of the state S and $[G(S)]$ is the disjoint union of reduced graphs that contain nodes. In this way, we obtain a sum of Laurent polynomials in A multiplying reduced graphs as the Manturov Parity Bracket. It is not hard to see that this bracket is invariant under regular isotopy and detour moves and that it behaves just like the usual bracket under the first Reidemeister move. However, the use of parity to make this bracket expand to graphical states gives it considerable extra power in some situations. For example, consider the Kishino diagram in Figure 10. We see that all the classical crossings in this knot are odd. Thus the parity bracket is just the graph obtained by putting nodes at each of these crossings. The resulting graph does not reduce under the graphical Reidemeister two moves, and so we conclude that the Kishino knot is non-trivial and non-classical. Since we can apply the parity bracket to a flat knot by taking $A = -1$, we see that this method shows that the Kishino flat is non-trivial.

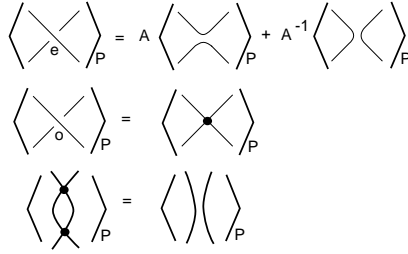


Figure 9: Parity bracket expansion.

In Figure 11 we illustrate the *Z-move* and the *graphical Z-move*. Two virtual knots or links that are related by a *Z-move* have the same standard bracket polynomial. This follows directly from our discussion in the previous section. We would like to analyze the structure of *Z-moves* using the parity bracket. In order to do

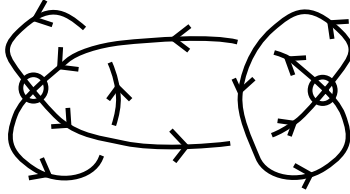


Figure 10: Kishino diagram.

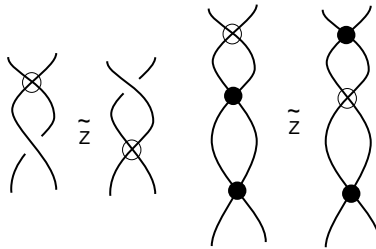


Figure 11: Z-move and graphical Z-move.

this we need a version of the parity bracket that is invariant under the Z -move. In order to accomplish this, we need to add a corresponding Z -move in the graphical reduction process for the parity bracket. This extra graphical reduction is indicated in Figure 11 where we show a graphical Z -move. The reader will note that graphs that are irreducible without the graphical Z -move can become reducible if we allow graphical Z -moves in the reduction process. For example, the graph associated with the Kishino knot is reducible under graphical Z -moves. However, there are examples of graphs that are not reducible under graphical Z -moves and Reidemeister two moves. An example of such a graph occurs in the parity bracket of the knot KS shown in Figure 12. This knot has one even classical crossing and four odd crossings. One smoothing of the even crossing yields a state that reduces to a loop with no graphical nodes, while the other smoothing yields a state that is irreducible even when the Z -move is allowed (see Figure 13). The upshot is that this knot KS is not Z -equivalent to any classical knot. Since one can verify that KS has unit Jones polynomial, this example is a counterexample to a conjecture of Fenn, Kauffman and Manturov [23] that suggested that a knot with unit Jones polynomial should be Z -equivalent to a classical knot. The existence of such counterexamples via parity was first pointed out by Vassily Manturov in 2009.

Parity is clearly an important theme in virtual knot theory and will figure in many future investigations of this subject. The type of construction that we have indicated for the bracket polynomial in this section can be varied and applied to other invariants. Furthermore the notion of describing a parity for crossings in a

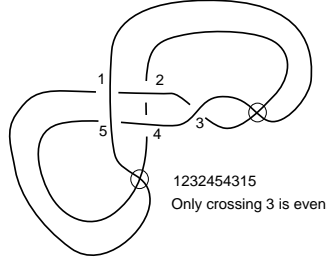


Figure 12: A knot KS with unit Jones polynomial.

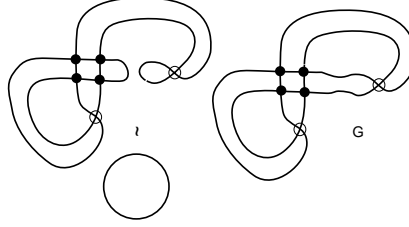


Figure 13: Parity bracket states for the knot KS .

diagram is also susceptible to generalization. For more on this theme the reader should consult [21, 22] and [24] for our original use of parity for another variant of the bracket polynomial.

5. Portraits of families of virtual KL s

Recursive and general formulas for the reduced relative Tutte polynomials can be computed for different families of virtual KL s, and from them we obtain Jones polynomials and Kauffman bracket polynomials of the considered families of virtual KL s.

Obtained results can be used to study properties of the reduced relative Tutte polynomials of virtual KL s and zeros of Kauffman bracket polynomials and Jones polynomials. The plot of zeros of Jones polynomials of virtual KL family is specific to the family and will be referred to as the "portrait of a virtual link family".

The portrait of the virtual link family $(i, 1^p)(1^q)$ ($1 \leq p \leq 20$, $2 \leq q \leq 20$) is shown in Fig. 14.

Portraits of families of virtual KL s obtained for different choices of signs of parameters are compared in Figs. 15 and 16. Fig. 15 is the portrait of the family $(1^p)i(1^q)$ for $2 \leq p \leq 20$, $2 \leq q \leq 20$, and the Fig. 16 corresponds to the family $(1^p)i((-1)^q)$ for $p, q \in \mathbb{Z}^+$. More detailed results of this kind will be given in the

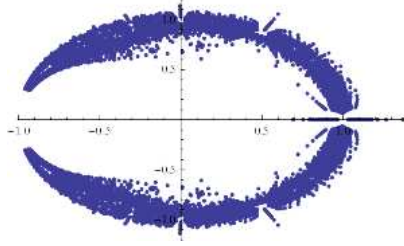


Figure 14: Plot of the zeroes of Jones polynomial for virtual KL family $(i, 1^p)(1^q)$ ($1 \leq p \leq 20$, $2 \leq q \leq 20$).

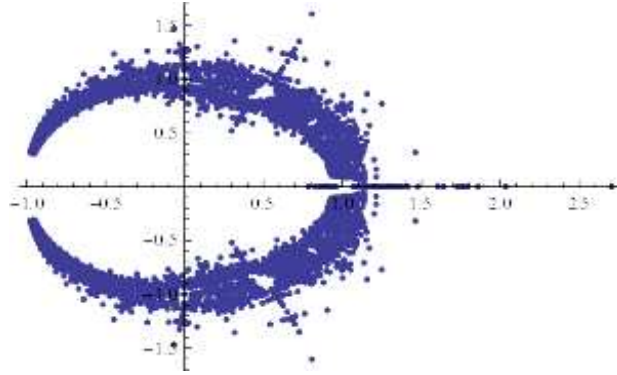


Figure 15: Plot of the zeroes of Jones polynomial for virtual KL family $(1^p)i(1^q)$ ($2 \leq p \leq 20$, $2 \leq q \leq 20$).

forthcoming paper.

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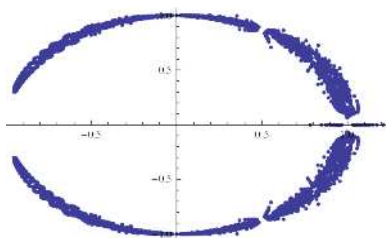


Figure 16: Plot of the zeroes of Jones polynomial for virtual KL family $(1^p) i((-1)^q)$ ($2 \leq p \leq 20$, $2 \leq q \leq 20$).

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